NUMERICAL DESCRIPTION OF WORKING PARAMETERS FOR HUMAN ELLIPTICAL HIP JOINT

SUMMARY

In 1971 Cathcart proposed the use of an elliptical head for the femoral endoprosthesis, based on his anatomical and clinical studies, which have proved that the normal femoral head is elliptical and not spherical. It was decided to use the an elliptical head of endoprosthesis because it improves the lubrication of acetabular cartilage and better distributes the loads.

Key-words:
Lubrication, human elliptical hip joint, working parameters

Słowa-kluczowe:
Smarowanie, eliptyczny staw biodrowy człowieka, parametry pracy

* University of Szczecin, Mathematical and Physical Department, Institute of Mathematics, Wielkopolska 15, 70-451 Szczecin, phone: (091) 433-12-69, fax: (091) 433-58-14, e-mail: czajko@wmf.univ.szczecin.pl
The numerical analysis of pressure, carrying capacity and compressive stresses are presented in this paper. It is taken into account the synovial unsymmetrical and stationary fluid flow in human elliptical hip joint. The paper shows the particular Reynolds equation for pressure distribution in hydrodynamic lubrication problem for human elliptical hip joint.

INTRODUCTION

Let us review what was happened in the presented below references. Dowson D. gives some basic biomechanics and biotribology of human synovial joints and joint replacements in the papers [L. 4–5]. Mow V.C. also presents the basic biomechanics of diarthrodinal human joints in the papers [L. 6–7]. Pipino F. and Molfetta L. describe the clinical and radiographic study of Cathcart elliptical orthocentric femoral prosthesis in the papers [L. 8–9]. These kind of elliptical prosthesis were firstly used in 1971. Wierzcholski K. was presented the mathematical and numerical model for working parameters in human hip joint for unsymmetrical and stationary fluid flow in the paper [L. 10] (pressure, capacity – spherical hip), for unsymmetrical and unstationary flow in the paper [L. 11] (pressure – spherical hip) and unsymmetrical and stationary flow caused not only by rotation but also by squeezing in the paper [L. 12] (pressure – spherical hip). The Autor of described a problem of working parameters in the human hip joint for stationary and unsymmetrical synovial fluid flow in the papers [L. 2] (pressure, capacity, stress – elliptical endoprosthesis) and [L. 3] (pressure, capacity, stress – elliptical hip joint, parameters: \( \omega = 0.5, \eta = 0.25 \) for normal and \( \omega = 0.4, \eta = 0.025 \) for pathological) and also for symmetrical fluid flow in the paper [L. 1] (pressure - elliptical hip joint).

The new element of this paper in compare with papers [L. 1–3], [L. 10–12] and also [L. 4–9] is the numerical analysis of pressure, capacity force and stresses made for elliptical human hip joint.

The numerical analysis of pressure, carrying capacity and compressive stresses shown for synovial fluid flow in elliptical hip joint are presented in this paper. It is taken into account the stationary, unsymmetrical and isothermal flow of incompressible synovial fluid. Moreover only the circumferential the rotational motion of femoral head causes the fluid flow. In this model are considered the changeable gap height of hip joint and changeable synovial fluid viscosity. The gap height changes only in
the limits of lubrication. The rotary motion of the elliptical femoral head it is possible only in the circumferential direction and also in the lubrication limits. It is neglected the roughness and irregularities occurring on the surface of the hip head and acetabulum. Fluid density is constant. It is taken the elliptical co-ordinates system (ϕ, r, ζ) – in circumference, radial and meridional direction respectively (Fig. 1). The components of the synovial fluid velocity vector \( \mathbf{v} = [V_\phi, V_r, V_\zeta] \) are shown on Fig. 2 and pressure localisation is presented on Fig. 3.

**Fig. 1. Elliptical system**
Rys. 1. Układ eliptyczny

**Fig. 2. Components of vector \( \mathbf{v} \)**
Rys. 2. Współrzędne wektora \( \mathbf{v} \)

**Fig. 3. Pressure localisation**
Rys. 3 Lokalizacja ciśnienia

**MATHEMATICAL DESCRIPTION**
For axially unsymmetrical and stationary synovial fluid flow, pressure function depends on \( \phi, \zeta \) and dynamic viscosity \( \eta_p \) of synovial fluid depends on \( \phi, r \) and \( \zeta \). The gap height \( \varepsilon \) may be a function of variable \( \phi \) and
ζ. The elliptical surfaces in hip joint create a gap where is synovial fluid. The only circumferential rotation of elliptical femoral head with angular velocity \( \omega \) causes a flow of synovial fluid in biobearing gap.

If we neglect the inertia and centrifugal forces, and then after layer boundary simplifications in general equations, we obtain Reynolds equation for pressure function \( p(\phi, \zeta) \) in the elliptical co-ordinate system. This equation has the following form:

\[
\frac{\partial}{\partial \phi} \left( \frac{\varepsilon^3}{\eta_0} \frac{\partial p}{\partial \phi} \right) + a \left[ \sqrt{ \cos^2 \left( \frac{\zeta}{a} \right) + \frac{b^2}{a^2} \sin^2 \left( \frac{\zeta}{a} \right) } \right]^{-1} \sin \left( \frac{\zeta}{a} \right) \times
\frac{\partial}{\partial \zeta} \left[ \frac{\varepsilon^3 a}{\eta_0} \left[ \sqrt{ \cos^2 \left( \frac{\zeta}{a} \right) + \frac{b^2}{a^2} \sin^2 \left( \frac{\zeta}{a} \right) } \right]^{-1} \frac{\partial p}{\partial \zeta} \sin \left( \frac{\zeta}{a} \right) \right] = 6 a^2 \omega \frac{\partial}{\partial \phi} \sin^2 \left( \frac{\zeta}{a} \right). \tag{1}
\]

\[
0 < \phi < 2\pi c_1, \quad 0 < c_1 < 1, \quad \frac{\pi a}{8} < \zeta < \frac{\pi a}{2}, \quad \zeta_1 = \frac{\zeta}{a}, \quad a < b \tag{2}
\]

where the symbol \( \omega \) means the angular velocity of femoral head in [1/s], \( \eta_0 \) – the characteristic synovial fluid viscosity in [Pa·s], \( a \) and \( b \) – the radii of elliptical femoral head in [m], \( \varepsilon \) – the gap height in [m], \( p \) – the pressure in [Pa].

A centre of elliptical femoral head is in the point \( O(0,0,0) \) and centre of elliptical cartilage is in the point \( O_1(x-\Delta \varepsilon, y-\Delta \varepsilon, z+\Delta \varepsilon) \) for hydrodynamic lubrication, which is caused only by head rotation.

On Fig. 3 is shown some location of pressure \( p(\phi, \zeta_1) \) on the head of hip joint with elliptical surface of lubrication \( \Omega(\phi, \zeta) \) in [m²]. In calculations is taken the values of atmospheric pressure \( p_{at} \) in [Pa]. Gap height calculated for elliptical femoral head has finally the following form:

\[
\varepsilon(\phi, \zeta_1) = \frac{a}{A} \sqrt{A^2 \sin^2 \zeta_1 + B^2 \cos^2 \zeta_1} \cdot f(\phi, \zeta_1) - \sqrt{a^2 \sin^2 \zeta_1 + b^2 \cos^2 \zeta_1} \tag{3}
\]

where \( f(\phi, \zeta_1) \equiv r_1 \) is the positive solution of the equation:

\[
[B^2a^2 \sin^2 \zeta_1 + A^2b^2 \cos^2 \zeta_1] r_1^2 + 2[B^2a(\Delta \varepsilon, \cos \phi \sin \zeta_1 + \Delta \varepsilon \sin \phi \cos \zeta_1) - A^2b \Delta \varepsilon, \cos \zeta_1] r_1 + \tag{4}
\]
\[ B^2[\Delta \varepsilon_1^2 + \Delta \varepsilon_2^2] + A^2 \Delta \varepsilon_3^2 - A^2 B^2 = 0, \]

\[ A \equiv a + D + \varepsilon_{\text{min}}, \quad B \equiv b + D + \varepsilon_{\text{min}}, \quad D \equiv \sqrt{\Delta \varepsilon_1^2 + \Delta \varepsilon_2^2 + \Delta \varepsilon_3^2} \]

where \( D \) means the eccentricity in [m] and \( r_1 \) is dimensionless radial direction in elliptical co-ordinate system and \( \varepsilon(\varphi, \xi) \) is changeable height of the gap in [m].

Total carrying capacity force \( C_{\text{tot}} \) in [N], area of lubrication \( S \) [cm\(^2\)] and compressive stress in [N/cm\(^2\)] on elliptical femoral head of the hip joint are obtained from formulae:

\[ C_{\text{tot}} \equiv \iint p(\varphi, \zeta) \, d\sigma(\varphi, \zeta), \quad (7) \]

\[ S \equiv \frac{\pi}{2} \left[ b^2 + \frac{a^2 b}{2\sqrt{b^2 - a^2}} \ln \left( \frac{b + \sqrt{b^2 - a^2}}{b - \sqrt{b^2 - a^2}} \right) \right] \cos \left( \frac{\pi}{8} \right) \quad a < b, \quad (8) \]

\[ \sigma \equiv \frac{C_{\text{tot}}}{S} \quad (9) \]

where symbol \( \iint […] \, d\sigma \) means the surface integral determined on the head of hip joint surface and the symbol \( d\sigma \) denotes area element in the double integral.

**NUMERICAL ANALYSIS**

In numerical analysis the symbols \( a, b \) mean the radii of elliptical head of hip joint in [cm], \( \omega \) – the real angular velocity of the head in [1/s], \( \eta \) – the right values of synovial fluid viscosity in [Pa\( \cdot \)s], \( \rho_{\text{max}} \) – the maximal calculated value of pressure in [Pa], \( \varepsilon_{\text{min}} \) – the minimum of the gap height in [\( \mu \text{m} \)], \( C_{\text{tot}} \) – the calculated total value of capacity in [N], \( S \) – the calculated surface region on femoral head in [cm\(^2\)] and \( \sigma \) – the calculated values of compressive stresses in [N/cm\(^2\)]. The symbols \( a, b, \omega, \eta, \varepsilon_{\text{min}} \) mean the based parameters and \( C_{\text{tot}}, \rho_{\text{max}}, S, \sigma \) are calculated parameters. Moreover in the numerical analysis using the formulae (1)-(9) are taken \( \Delta \varepsilon_1 = 2 \, \mu\text{m}, \Delta \varepsilon_2 = 2 \, \mu\text{m}, \Delta \varepsilon_3 = 2 \, \mu\text{m} \).

**Table 1. Parameters in numerical analysis of pressure for normal elliptical hip joint**
The results of pressure for normal elliptical hip joint are shown on Fig. 4–5.

Table 2. Parameters in analysis of pressure for pathological elliptical hip joint

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>1.02a = 2.448</td>
<td>0.25</td>
<td>0.25</td>
<td>1,504·10⁶</td>
<td>1</td>
<td>436</td>
<td>17,167</td>
<td>25,3975</td>
</tr>
<tr>
<td>2.4</td>
<td>1.03a = 2.472</td>
<td>0.25</td>
<td>0.25</td>
<td>1,480·10⁶</td>
<td>1</td>
<td>439</td>
<td>17,393</td>
<td>25,2400</td>
</tr>
<tr>
<td>Fig. 6</td>
<td>2.4</td>
<td>1.04a = 2.496</td>
<td>0.25</td>
<td>0.25</td>
<td>1,457·10⁶</td>
<td>1</td>
<td>441</td>
<td>17,621</td>
</tr>
<tr>
<td>Fig. 7</td>
<td>2.4</td>
<td>1.05a = 2.520</td>
<td>0.25</td>
<td>0.25</td>
<td>1,433·10⁶</td>
<td>1</td>
<td>443</td>
<td>17,849</td>
</tr>
</tbody>
</table>

The results of pressure for pathological elliptical hip joint are shown on Fig. 6-7.

Fig. 4. Pressure distribution for normal elliptical hip joint: a = 0.025; b = 1.04a [m]

Rys. 4. Rozkład ciśnienia dla zdrowego eliptycznego stawu biodrowego: a = 0.025; b = 1.04a [m]
Fig. 5. Pressure distribution for normal elliptical hip joint: $a = 0.025; b = 1.05a$ [m]

Rys. 5. Rozkład ciśnienia dla zdrowego eliptycznego stawu biodrowego: $a = 0.025; b = 1.05a$ [m]

Fig. 6. Pressure distribution for pathological elliptical hip joint: $a = 0.024; b = 1.04a$ [m]

Rys. 6. Rozkład ciśnienia dla patologicznego eliptycznego stawu biodrowego: $a = 0.024; b = 1.04a$ [m]
a=0.024 [m]  
b=1.05·a  
ω=0.25 [1/s]  
η=0.05 [Pas]  
p_{\text{max}} = 1.433 \cdot 10^6 [Pa]  
\varepsilon_{\text{min}} = 1 [\mu m]  
C_{\text{tot}} = 443 [N]  
Lubrication surface = 17.849 [cm^2]

Fig. 7. Pressure distribution for pathological elliptical hip joint: a = 0.024; b = 1.05a [m]

Rys. 7. Rozkład ciśnienia dla patologicznego eliptycznego stawu biodrowego: a = 0.024; b = 1.05a [m]

CONCLUSIONS
1. The analytical and numerical model of lubrication problem for human hip joint with elliptical co-operating surfaces allows to make the analysis of pressure, carrying capacity and compressive stresses.
2. For normal elliptical hip joint with based parameters a = 2.5 [cm], 2.550 [cm] < b < 2.625 [cm], ω = 0.75 [1/s], η = 0.2 [Pa·s], ε_{\text{min}} = 3 [\mu m] obtained in the numerical analysis the following values for pressure from 2,026 \cdot 10^6 [Pa] to 1,998 \cdot 10^6 [Pa], for total carrying capacity – from 916 [N] to 945 [N], for surface area from 18,627 [cm^2] to 19,368 [cm^2] and compressive stresses – from 49,1759 [N/cm^2] to 48,7918 [N/cm^2].
3. For pathological elliptical hip joint with based parameters a = 2.4 [cm], 2.448 [cm] < b < 2.520 [cm], ω = 0.25 [1/s], η = 0.25 [Pa·s], ε_{\text{min}} = 1 [\mu m] obtained in the numerical analysis the following values for pressure from 1,504 \cdot 10^6 [Pa] to 1,433 \cdot 10^6 [Pa], for total carrying capacity – from 436 [N] to 443 [N], for surface area from 17,167 [cm^2] to
17.849 \text{[cm}^2\text{]} \text{ and compressive stresses – from 25.3975 \text{[N/cm}^2\text{]} \text{ to 24.8193 \text{[N/cm}^2\text{].}}

4. The pressure, carrying capacity forces and compressive stresses values obtained for normal elliptical hip joint are smaller in compare with pressure, carrying capacity and compressive stresses values for pathological hip joint.

Acknowledgement: The author would like to thank Prof. zw. dr hab. inż. Krzysztof Wierczolski for his discussion during writing this paper. Author also thanks The State Committee for Scientific Research in Warsaw for his finance help at the realization of Grant No.8-T 11E-021-17.

REFERENCES


Streszczenie

Z wieloletnich badań i obserwacji klinicznych wynika, że głowa stawu biodrowego człowieka ma raczej kształt elipsoidalny niż sferyczny. Fakt ten uzasadnia stosowanie endoprotez tego stawu o zarysie głowy elipsoidalnym, jednak pod warunkiem zachowania własnej panewki (acetabulum). Stosowanie takich protez rozpoczął Cathcart już w 1971 roku [8]-[10].

W pracy zostały obliczone parametry pracy stawu biodrowego człowieka o eliptycznym zarysie współpracujących powierzchni. W modelu analitycznie przyjęto izotermiczny, stacjonarny i oświetleniowy przepływ cieczy synowialnej o nienewtonowskich właściwościach. Po oszacowaniu równań podstawowych otrzymano uproszczony układ równań różniczkowych cząstkowych, z którego uzyskano równanie Reynolds’a wyznaczające ciśnienie. Równanie to zostało rozwiązane numerycznie. W hydrodynamicznym problemie skarowania stawu biodrowego wyznaczono wartość maksymalną ciśnienia, wartość naprężenia, wartość siły nośnej.

Dla stawu normalnego przy parametrach: a=2,5 cm i 2,550 < b < 2,625 cm, \( \omega =0,75 \) 1/s, \( \eta=0,2 \) Pa-s, \( \varepsilon_{\text{min}}=3 \) µm uzyskano wartość ciśnienia od 2,026\( \cdot10^6 \) do 1,998\( \cdot10^6 \) Pa, siły nośnej – od 916 do 945 N, naprężeń – od 49,1759 do 48,7918 N/cm\(^2\) przy powierzchni od 18,627 do 19,368 cm\(^2\).

Natomiast dla stawu chorego (osteoartrotycznego) przy parametrach: a=2,4 cm i 2,448 < b < 2,520 cm, \( \omega =0,25 \) 1/s, \( \eta=0,25 \) Pa-s, \( \varepsilon_{\text{min}}=1 \) µm uzyskano wartość ciśnienia od 1,504\( \cdot10^6 \) do 1,433\( \cdot10^6 \) Pa,
sily nośnej – od 436 do 443 N, naprężeń – od 25,3975 do 24,8193 N/cm² przy powierzchni od 17,167 do 17,849 cm².

Wyniki te mogą przyczynić się do lepszej diagnostyki i terapii ortopedycznej ludzkiego stawu biodrowego oraz jego protez.